1037.Proposed by George Apostolopoulos, Messolonghi, Greece.

Let P be a point inside the triangle ABC and let D,E,F be the projections of P on the sides BC,CA, and AB, respectively. Prove that

$$\frac{PA+PB+PC}{(EF\cdot FD\cdot DE)^{1/3}} \geq 2\sqrt{3}$$

Solution by Arkady Alt , San Jose ,California, USA.

Let $R_a := PA, R_b := PB, R_c := PC$ and $a_p := EF, b_p := FD, c_p := DE$ (that is a_p,b_p,c_p are sidelengths of pedal triangle of point P). Then original inequality in the new notation becomes

(**1**) $$\frac{R_a+R_b+R_c}{(a_pb_pc_p)^{1/3}} \geq 2\sqrt{3}$$

Since quadrilateral $FAEP$ is cyclic (because $PF \perp AB$ and $PE \perp AC$) and R_a is diameter of circumcircle of quadrilateral $FAEP$ then $\frac{a_p}{R_a} = \sin A = \frac{a}{2R}$ and, similarly, $\frac{b_p}{R_b} = \sin B = \frac{b}{2R}$, $\frac{c_p}{R_c} = \sin C = \frac{c}{2R}$ and inequality (**1**) can be rewritten as $\dfrac{R_a+R_b+R_c}{\left(\frac{aR_a}{2R}\cdot\frac{bR_b}{2R}\cdot\frac{cR_c}{2R}\right)^{1/3}} \geq 2\sqrt{3} \iff \dfrac{2R(R_a+R_b+R_c)}{(aR_a\cdot bR_b\cdot cR_c)^{1/3}} \geq 2\sqrt{3} \iff$

$\dfrac{R(R_a+R_b+R_c)}{\sqrt[3]{R_aR_bR_c}} \geq \sqrt{3}\sqrt[3]{abc} \iff \sum_{cyc}\sqrt[3]{\dfrac{R_a^2}{R_bR_c}} \geq \dfrac{\sqrt{3}}{R}\sqrt[3]{abc}$ or $\sum_{cyc}\sqrt[3]{\dfrac{R_a^2}{R_bR_c}} \geq \sqrt{3}\sqrt[3]{\dfrac{sr}{R^2}}$.

Or, inequality (**1**) can be rewritten as $\dfrac{R_a+R_b+R_c}{(R_a\sin A\cdot R_b\sin B\cdot R_c\sin C)^{1/3}} \geq 2\sqrt{3} \iff$

$\dfrac{R_a+R_b+R_c}{(R_aR_bR_c)^{1/3}} \geq 2\sqrt{3}(\sin A\sin B\cdot\sin C)^{1/3}$.

Since $\dfrac{R_a+R_b+R_c}{(R_aR_bR_c)^{1/3}} \geq 3$ suffice to prove that $3 \geq 2\sqrt{3}(\sin A\sin B\cdot\sin C)^{1/3} \iff$

$\dfrac{\sqrt{3}}{2} \geq (\sin A\sin B\cdot\sin C)^{1/3}$.

We have $\dfrac{\sin A+\sin B+\sin C}{3} \leq \dfrac{\sqrt{3}}{2}$ (because for $\sin x$ which is concave down on $[0,\pi]$ by Jensen's Inequality holds $\dfrac{\sin A+\sin B+\sin C}{3} \leq \sin\dfrac{A+B+C}{3} = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$)

and by AM-GM $(\sin A\sin B\cdot\sin C)^{1/3} \leq \dfrac{\sin A+\sin B+\sin C}{3}$.

(Another way to prove inequality $\sin A+\sin B+\sin C \leq \dfrac{3\sqrt{3}}{2}$.

First note that for any $x,y \in [0,\pi]$ holds inequality $\sin x+\sin y \leq 2\sin\dfrac{x+y}{2}$.

Indeed, $\sin x+\sin y = 2\sin\dfrac{x+y}{2}\cos\dfrac{x-y}{2} \leq 2\sin\dfrac{x+y}{2}$ because

$\dfrac{x+y}{2} \in [0,\pi]$ and $\dfrac{x-y}{2} \in [-\pi/2,\pi/2]$.

Using inequality $\sin x+\sin y \leq 2\sin\dfrac{x+y}{2}$ we obtain

$$\sin A+\sin B+\sin C+\sin\frac{\pi}{3} \leq 2\sin\frac{A+B}{2}+2\sin\frac{C+\frac{\pi}{3}}{2} \leq 4\sin\frac{\frac{A+B}{2}+\frac{C+\frac{\pi}{3}}{2}}{2} =$$

$$4\sin\frac{\pi+\frac{\pi}{3}}{4} = 4\cdot\sin\frac{\pi}{3} = 2\sqrt{3} \Rightarrow \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}).$$